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Relations among operator orders and operator inequalities

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1 The Furuta inequality and the chaotic order

In what follows, an operator means a bounded linear operator on a Hilbert space H and is denoted by a capital letter. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

We start this report with introduction of the following order preserving operator inequalities.

Theorem F (Furuta inequality [5]).

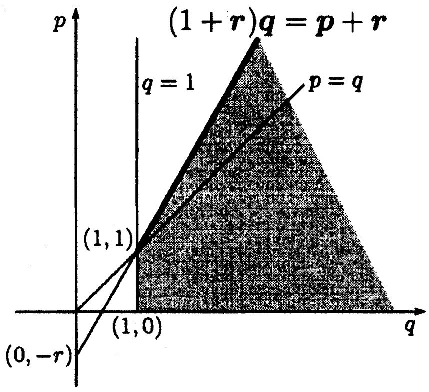
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Theorem F yields the famous Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*" by putting $r = 0$ in (i) or (ii) of Theorem F. An elementary one-page proof of Theorem F was given in [6]. It was shown in [15] that the domain of the parameters is the best possible in Theorem F.

The order defined by $\log A \geq \log B$ for $A, B > 0$ is called the chaotic order. The chaotic order is weaker than the usual order since $\log \cdot$ is an operator monotone function. The following characterization of the chaotic order is an application of Theorem F and an extension of a result in [1].

Theorem 1.A ([3][7]). *Let $A, B > 0$. Then the following are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all* $p > 0$ *and* $r > 0$.

(iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p > 0$ *and* $r > 0$.

We remark the correspondence of Theorem 1.A to the essential part of Theorem F: $A \geq B \geq 0$ *ensures*

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r} \quad \textit{and} \quad A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

for all $p > 1$ *and* $r > 0$. Another simple proof of Theorem 1.A was given in [17]. It was shown in [18] that the domain of the parameters is the best possible in Theorem 1.A. It can be proved by the following Lemma F that

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Longleftrightarrow A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}} \qquad (*)$$

holds for $A, B > 0$ and $p, r > 0$.

Lemma F ([9]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}} B^* B A^{\frac{1}{2}})^{\lambda-1} A^{\frac{1}{2}} B^*$$

holds for any real number λ.

It was shown in [14] that similar relations to $(*)$ hold even if A and B are not invertible.

Theorem 1.B ([14]). *Let $A, B \geq 0$. Then for each $p > 0$ and $r > 0$, the following hold:*

(i) *If* $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, *then* $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.

(ii) *If* $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and* $N(A) \subseteq N(B)$, *then* $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.

2 Operator inequalities related to the relative operator entropy

The relative operator entropy was defined in [2] as $S(A \mid B) = A^{\frac{1}{2}} \log(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$ for $A, B > 0$. We remark that $S(A \mid I) = -A \log A$ is the operator entropy. In case $p, r > 0$,

$$A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} \Longrightarrow \log A^{p+r} \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})$$

holds for $A, B > 0$, so that (iii) $\Longrightarrow$ (i) of the following Theorem 2.A is an extension of (iii) $\Longrightarrow$ (i) of Theorem 1.A.

Theorem 2.A ([8]). *Let $A, B > 0$. Then the following are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all $p > 0$ and $r > 0$.*

(iii) $\log A^{p+r} \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})$ *for all $p > 0$ and $r > 0$.*

(iv) $S(A^{-r} \mid A^p) \geq S(A^{-r} \mid B^p)$ *for all $p > 0$ and $r > 0$.*

Here we consider the case $p > 0 > r$. We obtain the following result by applying Theorem 1.A.

Proposition 2.1. *Let $A, B > 0$ and $p > 0$.*

(i) *In case $s > -p$,* $\log A^{p+s} \geq \log(A^{\frac{s}{2}} B^p A^{\frac{s}{2}}) \Longleftrightarrow A^{-s+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{-s+r}{p+r}}$ *for all $r > s$.*

(ii) *In case $s < -p$,* $\log A^{p+s} \geq \log(A^{\frac{s}{2}} B^p A^{\frac{s}{2}}) \Longleftrightarrow A^{-s+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{-s+r}{p+r}}$ *for all $r < s$.*

The following is an immediate corollary of Proposition 2.1.

Corollary 2.2. *Let $A, B > 0$ and $p > t > 0$.*

$$A^p \geq B^p \Longrightarrow \log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \Longrightarrow A^t \geq B^t.$$

Corollary 2.2 corresponds to the case $\beta \nearrow t$ of the following result.

Proposition 2.B ([12]). *Let $A, B > 0$ and $p > t > \beta \geq 0$.*

$$A^\gamma \geq B^\gamma \Longrightarrow A^{t-\beta} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{t-\beta}{p-t}} \Longrightarrow A^\delta \geq B^\delta,$$

where $\gamma = \max\{2t - \beta, p\}$ and $\delta = \min\{2t - \beta, p\}$.

Proof of Proposition 2.1. $\log A^{p+s} \geq \log(A^{\frac{s}{2}} B^p A^{\frac{s}{2}})$ implies

$$A^{(p+s)r_1} \geq \left\{A^{\frac{(p+s)r_1}{2}} (A^{\frac{s}{2}} B^p A^{\frac{s}{2}}) A^{\frac{(p+s)r_1}{2}}\right\}^{\frac{r_1}{1+r_1}}$$

for $r_1 = \frac{-s+r}{p+s} > 0$ by Theorem 1.A, then we have ($\Rightarrow$). ($\Leftarrow$) is obtained by taking the logarithms of both sides of $A^{-s+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{-s+r}{p+r}}$ and letting $r \to s$. $\square$

Proof of Corollary 2.2. The first implication is obvious since $\log \cdot$ is operator monotone, and the second is obtained by putting $s = -t < 0$ and $r = 0$ in (i) of Proposition 2.1. $\square$

We can summarize relations among orders and the inequality $\log A^{p+r} \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})$ as follows.

(i) In case $p, r > 0$,

$$\begin{tikzcd}[row sep=small] A^p \geq B^p \arrow[dr, Rightarrow] & & \\ & \log A \geq \log B \arrow[r, Rightarrow] & \log A^{p+r} \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}). \\ A^r \geq B^r \arrow[ur, Rightarrow] & & \end{tikzcd}$$

(ii) In case $p > t > 0$,

$$A^p \geq B^p \Longrightarrow \log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \Longrightarrow A^t \geq B^t \Longrightarrow \log A \geq \log B.$$

(iii) In case $t > p > 0$,

$$\begin{tikzcd}[row sep=small] & & \log A \geq \log B \\ A^t \geq B^t \arrow[r, Rightarrow] & A^p \geq B^p \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & & \log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}). \end{tikzcd}$$

We obtain the following result on the best possibility of Corollary 2.2.

Proposition 2.3.

(i) *Let $p > q > 0$ and $t > 0$. Then there exist $A, B > 0$ such that*

$$A^q \geq B^q \quad \textit{and} \quad \log A^{p-t} \not\geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}).$$

(ii) *Let $p > t > 0$ and $q > t$. Then there exist $A, B > 0$ such that*

$$\log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \quad \textit{and} \quad A^q \not\geq B^q.$$

Proposition 2.3 can be proved by applying the following results.

Theorem 2.C ([16]). *Let $p > 1$ and $t > 0$. If $\alpha > 0$, then there exist $A, B > 0$ such that*

$$A \geq B \quad \textit{and} \quad A^{(p-t)\alpha} \not\geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\alpha}.$$

Theorem 2.D ([18]). *Let $p > 0$ and $r > 0$. If $\alpha > 1$, then there exist $A, B > 0$ such that*

$$\log A \geq \log B \quad \textit{and} \quad A^{r\alpha} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r\alpha}{p+r}}.$$

Proof of Proposition 2.3.
Proof of (i). The case $p = t$ can be proved easily since $0 \geq \log(A^{\frac{-p}{2}} B^p A^{\frac{-p}{2}})$ is equivalent to $A^p \geq B^p$. In case $p > t$, there exist $A_1, B_1 > 0$ such that

$$A_1 \geq B_1 \quad \text{and} \quad A_1^{(p_1-t_1)\alpha} \not\geq (A_1^{\frac{-t_1}{2}} B_1^{p_1} A_1^{\frac{-t_1}{2}})^{\alpha}$$

for $p_1 = \frac{p}{q} > 1$, $t_1 = \frac{t}{2q} > 0$ and $\alpha = \frac{t}{2p-t} > 0$ by Theorem 2.C. Put $A = A_1^{\frac{1}{q}}$, $B = B_1^{\frac{1}{q}}$ and $r_1 = \frac{t}{2(p-t)} > 0$, then we have

$$A^q \geq B^q \quad \text{and} \quad A^{(p-t)r_1} \not\geq \left\{A^{\frac{(p-t)r_1}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) A^{\frac{(p-t)r_1}{2}}\right\}^{\frac{r_1}{1+r_1}},$$

so that $\log A^{p-t} \not\geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})$ by Theorem 1.A.

In case $p < t$, there exist $A_1, B_1 > 0$ such that

$$A_1 \geq B_1 \quad \text{and} \quad A_1^{(p_1-t_1)\alpha} \not\geq (A_1^{\frac{-t_1}{2}} B_1^{p_1} A_1^{\frac{-t_1}{2}})^{\alpha}$$

for $p_1 = \frac{p}{q} > 1$, $t_1 = \frac{2t}{q} > 0$ and $\alpha = \frac{-t}{p-2t} > 0$ by Theorem 2.C. Put $A = A_1^{\frac{1}{q}}$, $B = B_1^{\frac{1}{q}}$ and $r_1 = \frac{-t}{p-t} > 0$, then we have

$$A^q \geq B^q \quad \text{and} \quad A^{(p-t)r_1} \not\geq \left\{A^{\frac{(p-t)r_1}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})A^{\frac{(p-t)r_1}{2}}\right\}^{\frac{r_1}{1+r_1}},$$

so that $\log A^{p-t} \not\geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})$ by Theorem 1.A.

Proof of (ii). There exist $A_1, B_1 > 0$ such that

$$\log A_1 \geq \log B_1 \quad \text{and} \quad A_1^{r_1\alpha} \not\geq (A_1^{\frac{r_1}{2}} B_1 A_1^{\frac{r_1}{2}})^{\frac{r_1\alpha}{1+r_1}}$$

for $r_1 = \frac{t}{p-t} > 0$ and $\alpha = \frac{q}{t} > 1$ by Theorem 2.D, then we have the desired conclusion by putting $A = A_1^{\frac{1}{p-t}}$ and $B = (A_1^{\frac{t}{2(p-t)}} B_1 A^{\frac{t}{2(p-t)}})^{\frac{1}{p}}$, that is, $A_1 = A^{p-t}$ and $B_1 = A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}$. □

We obtain the following result by applying (i) of Proposition 2.3.

Theorem 2.4. *Let $p > t$, $s > 1$ and $r < 0$. Then there exist $A, B > 0$ such that*

$$A^p \geq B^p \quad \textit{and} \quad \log A^{(p-t)s+r} \not\geq \log\{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}.$$

Proof. There exist $A_1, B_1 > 0$ such that

$$A_1 \geq B_1 \quad \text{and} \quad \log A_1^{s-t_1} \not\geq \log(A_1^{\frac{-t_1}{2}} B_1^s A_1^{\frac{-t_1}{2}}).$$

for $t_1 = \frac{-r}{p-t} > 0$ by (i) of Proposition 2.3, then we have the desired conclusion by putting $A = A_1^{\frac{1}{p-t}}$ and $B = (A_1^{\frac{t}{2(p-t)}} B_1 A^{\frac{t}{2(p-t)}})^{\frac{1}{p}}$, that is, $A_1 = A^{p-t}$ and $B_1 = A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}$. □

It turns out by Theorem 2.4 that the generalized Furuta inequality ([9])

"*$A \geq B \geq 0$ with $A > 0 \Longrightarrow A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$*

for $p \geq 1$, $t \in [0,1]$, $s \geq 1$ and $r \geq t$"

is not valid for $p \geq 1$, $p > t$, $s > 1$ and $r < 0$.

3 Operator inequalities in a characterization of the chaotic order

The following relation holds between the inequalities in Theorem 1.A for $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$. In fact, this relation can be proved by Theorem F and Lemma F in case A and B are invertible, and by Theorem 1.B in case they are not invertible.

Proposition 3.A ([11][14]). *Let $A, B \geq 0$, $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$.*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \geq B^{r_2}.$$

Here we consider the case $p_1 > p_2$ or $r_1 > r_2$ in Proposition 3.A. In case A and B are not invertible, the following was shown in the proof of [13, Theorems 5, 6].

Theorem 3.B ([13]). *Let $p_1 > 0$ and $r_1 > 0$. Then there exist $A, B \geq 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \text{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}$$

for all $p_2 > 0$ and $r_2 > 0$ such that $p_1 > p_2$.

In case A and B are invertible, the following was given as a concrete example for $p_1 = r_1 = 2$ and $p_2 = r_2 = 1$.

Example 3.C ([4][10]).

Let $A = \begin{pmatrix} 17 & 7 \\ 7 & 5 \end{pmatrix}^2$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^2$. Then $A, B > 0$, $(BA^2B)^{\frac{1}{2}} \geq B^2$ and $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$.

We obtain the following result by applying Proposition 3.A and Example 3.C.

Theorem 3.1. *Let $p_1 > p_2 > 0$ and $r_1 > r_2 > 0$. Then there exist $A, B > 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \text{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}.$$

It turns out by Lemma F that A and B in Theorem 3.1 also satisfy

$$A^{p_1} \geq (A^{\frac{p_1}{2}} B^{r_1} A^{\frac{p_1}{2}})^{\frac{p_1}{p_1+r_1}} \quad \text{and} \quad A^{p_2} \not\geq (A^{\frac{p_2}{2}} B^{r_2} A^{\frac{p_2}{2}})^{\frac{p_2}{p_2+r_2}}.$$

Proof. Assume that the following holds for $A, B > 0$:

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \geq B^{r_2}. \tag{3.1}$$

By Proposition 3.A and (3.1), we have

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{\theta r_1}{2}} A^{\theta p_1} B^{\frac{\theta r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{\theta r_1}, \tag{3.2}$$

where $\theta = \max\{\frac{p_2}{p_1}, \frac{r_2}{r_1}\} < 1$. Let n be an integer such that $\theta^n \leq \min\{\frac{p_1}{2r_1}, \frac{r_1}{2p_1}\}$. By applying (3.2) n times, we have

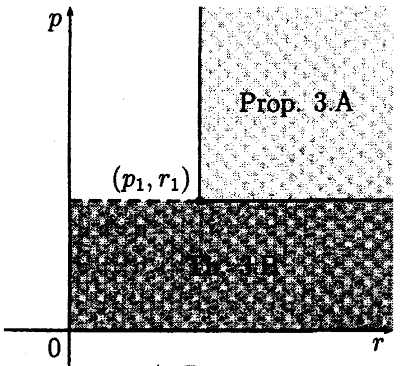
$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{\theta^n r_1}{2}} A^{\theta^n p_1} B^{\frac{\theta^n r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{\theta^n r_1}. \tag{3.3}$$

By Proposition 3.A and (3.3), we have

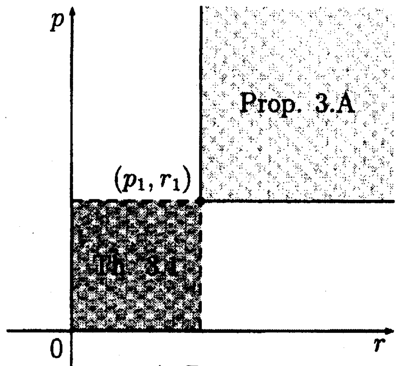
$$(B^{\frac{t}{2}} A^t B^{\frac{t}{2}})^{\frac{1}{2}} \geq B^t \Longrightarrow (B^{\frac{t}{4}} A^{\frac{t}{2}} B^{\frac{t}{4}})^{\frac{1}{2}} \geq B^{\frac{t}{2}}, \tag{3.4}$$

where $t = \min\{p_1, r_1\}$. The proof is complete since (3.4) contradict to Example 3.C. □

The domains of (p_2, r_2) in Proposition 3.A, Theorem 3.B and Theorem 3.1 are as in the following figures.



$A, B \geq 0$



$A, B > 0$

The following remains an open problem which corresponds to the case A and B are invertible in Theorem 3.B.

Conjecture 3.2. *Let $p_1 > 0$ and $r_1 > 0$. Then there exist $A, B > 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \textit{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}$$

for all $p_2 > 0$ and $r_2 > 0$ such that $p_1 > p_2$.

The following follows from Conjecture 3.2 by Lemma F since A and B are invertible in Conjecture 3.2.

Conjecture 3.3. *Let $p_1 > 0$ and $r_1 > 0$. Then there exist $A, B > 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \textit{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}$$

for all $p_2 > 0$ and $r_2 > 0$ such that $p_1 > p_2$ or $r_1 > r_2$.

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